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Adjoint Method for Design Sensitivity Analysis of Multiple Eigenvalues and Associated Eigenvectors

Tae Hee Lee*
Hanyang University, Seoul 133-791, Republic of Korea

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The adjoint method for design sensitivity analysis of multiple eigenvalues and their associated eigenvectors is proposed for real symmetric structural eigenvalue problems, where we restrict that the design sensitivity coefficients of the multiple eigenvalues are distinct. The developed adjoint method is required to compute adjoint variables from the simultaneous linear system equation, the so-called adjoint equation, without the linear combination of eigenvectors or the second-order derivatives of the eigenvalue problem. Once the adjoint variables are evaluated, the sensitivity analysis of the multiple eigenvalues and their associated eigenvectors can be computed directly. In this way, the design sensitivity analysis of eigenvectors can be obtained by using the eigenvalues and their eigenvectors of the mode being differentiated only. To verify the proposed method, analytical and numerical examples are demonstrated. This can have considerable impact on computer implementation of the method in the design sensitivity analysis of the eigenvalue problem needed for practical application.

I. Introduction

EIGENPROBLEMS are commonly considered in structural stability, noise, and vibration analyses. Design sensitivity analysis computes the rate of change of response-dependent function with respect to the change of design variables. Design sensitivity analysis is an essential step to improve systematically the existing design and to optimize a system with the aid of gradient-based optimization. Therefore an efficient and accurate method for design sensitivity analysis is necessary for diverse design optimizations.

Fox and Kapoor have developed a general technique to compute design sensitivity of eigenvalues for symmetric matrices [1]. Plaut and Husseyin [2], Rudisill [3], and Godoy et al. [4] have developed a formula for second-order design sensitivity analysis of eigenvalues. For the design sensitivity analysis of eigenvectors, the modal method [1,5], the modified modal method [6], and Nelson's method [7] are often used to calculate the derivatives of eigenvectors. A comparison of the above methods for calculating eigenvector derivatives has been carried out [8]. The modal method approximates the derivatives of eigenvectors as a linear combination of all eigenvectors. The modified modal method is developed to reduce the number of eigenvectors needed to represent the design derivative by including an additional term in the linear combination of eigenvectors. The modal and modified modal methods can be computationally expensive if large numbers of eigenvectors are needed to represent accurately the derivatives of eigenvectors. Nelson's method is a direct differentiation method for calculating eigenvector derivatives, where it requires only the eigenvalue and eigenvectors for the mode being differentiated. Recently an adjoint method for calculating derivatives of distinct eigenvalue and their associated eigenvectors is suggested, where an adjoint equation of simultaneous linear system equations requires only the eigenvalues and eigenvectors for the mode being differentiated [9].

For many typical structures, there exist multiple or nearly identical eigenvalues due to structural symmetry. In the case of multiple eigenvalues, their associated eigenvectors are generally not differentiable. Therefore the methods for the design sensitivity analysis of distinct eigenvalues will no longer be valid. The eigenvectors corresponding to multiple eigenvalues have a great deal of uncertainty compared with those associated with distinct eigenvalues because any linear combination of eigenvectors is also a valid eigenvector.

For design sensitivity analysis of multiple eigenvalues, Ojalvo [10] has provided an equation to calculate the derivatives of multiple eigenvalues and their associated eigenvectors by extending Nelson's method when the derivatives of the eigenvalues are distinct. Mills-Curran [11] and Dailey [12] have independently modified and corrected Ojalvo's method by differentiating the eigenvalue equation twice, that is, the second order design sensitivity analysis of eigenvalue. Friswell [13] has also extended the Nelson's method to handle the multiple eigenvalue and their associated eigenvectors. Juang et al. [14], Bernard and Bronowicki [15], and Lee and Jung [16] have improved the modal method to evaluate the multiple eigenvalue and eigenvector sensitivity. Calculation of sensitivities for multiple eigenvalues has been obtained without using associated eigenvectors for the structural problem [17,18].

The proposed new method for design sensitivity analysis of multiple eigenvalues and their associated eigenvectors is an adjoint method that requires evaluation of the adjoint variables from the simultaneous system equation, the so-called adjoint equation with added side constraints. First we define the augmented function that consists of the response function and eigenvalue equations. Then the design derivatives of the response function are represented as an explicit design variation of the augmented function and an adjoint equation is obtained by requiring the implicit design variation of the augmented function to vanish. It is important to note that the adjoint equation is composed of eigenvalues and their associated eigenvectors of the mode being differentiated. Once we evaluate the adjoint variables, design sensitivity analysis of multiple eigenvalues and their associated eigenvectors can be computed directly.

To verify the proposed method, an analytical example of a two-degree-of-freedom spring-mass system, planar grillage structure, and a finite element example of cantilever beam are included. Further the developed method can be easily implemented into a commercial finite element program to carry out the design sensitivity analysis of eigenproblems needed for practical applications.

II. Definition of Problem

Structural modal analysis and linear buckling analysis lead to the generalized eigenvalue problem as follows:

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*Professor, School of Mechanical Engineering, Sungdong-ku; thlee@hanyang.ac.kr. Senior Member AIAA.

$$\boldsymbol{KX} = \boldsymbol{MX\Lambda} \qquad \boldsymbol{K}, \boldsymbol{M} \in R^{n\times n} \tag{1}$$

where $\boldsymbol{K}$ and $\boldsymbol{M}$ represent the stiffness matrix and the mass matrix in vibration analysis, respectively. $\boldsymbol{M}$ is positive definite matrix and $\boldsymbol{K}$ is positive definite or at least positive semidefinite matrix. The eigenvalue matrix arranged in ascending order is represented as

$$\boldsymbol{\Lambda} = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \tag{2}$$

The associated eigenvector matrix whose column denotes the associated eigenvector can be arranged as follows:

$$\boldsymbol{X} = [\boldsymbol{x}_1 \quad \boldsymbol{x}_2 \quad \cdots \quad \boldsymbol{x}_n], \qquad \boldsymbol{x}_i \in R^n \quad \text{and} \quad \boldsymbol{X} \in R^{n\times n} \tag{3}$$

Because the eigenvectors (mode shape) is often normalized with respect to a symmetric positive definite matrix, we take M-orthonormality condition as follows:

$$\boldsymbol{X}^T\boldsymbol{MX} = \boldsymbol{I} \tag{4}$$

where the right superscript T denotes the transposition of a matrix and $\boldsymbol{I}$ represents the identity matrix.

For the case of the multiple eigenvalues of multiplicity m, the eigenvalue problem can be defined as follows:

$$\boldsymbol{KX} = \boldsymbol{MX\Lambda} \qquad \boldsymbol{\Lambda} = \lambda\boldsymbol{I} \in R^{m\times m} \tag{5}$$

where $\lambda \in R$ is an eigenvalue of multiplicity m, that is, it is the eigenvalue for the space spanned by the columns of $\boldsymbol{X} \in R^{n\times m}$ that is called the associated eigenvectors.

III. Design Sensitivity Analysis of Eigenproblem

A. Nelson's Method

To find the derivatives of the ith eigenvalue and its associated eigenvectors in the case of distinct eigenvalue, we differentiate Eqs. (1) and (4) with respect to a design variable b as follows:

$$(\boldsymbol{K} - \lambda_i\boldsymbol{M})\frac{d\boldsymbol{x}_i}{db} = \left[\frac{d\lambda_i}{db}\boldsymbol{M} - \frac{\partial \boldsymbol{K}}{\partial b} + \lambda_i\frac{\partial \boldsymbol{M}}{\partial b}\right]\boldsymbol{x}_i \tag{6}$$

$$\boldsymbol{x}_i^T\boldsymbol{M}\frac{d\boldsymbol{x}_i}{db} = -\frac{1}{2}\boldsymbol{x}_i^T\frac{\partial \boldsymbol{M}}{\partial b}\boldsymbol{x}_i \tag{7}$$

Note that the design variable must be independent. The coefficient matrix $(\boldsymbol{K} - \lambda_i\boldsymbol{M})$ of Eq. (6) has rank $(n-1)$ because all eigenvalues are assumed to be distinct in this method. Thus the linear combination of particular and homogeneous solutions to Eq. (6) becomes the solution to Eq. (6) as follows:

$$\frac{d\boldsymbol{x}_i}{db} = \boldsymbol{v}_i + c\boldsymbol{x}_i \tag{8}$$

Setting an element of $\boldsymbol{v}_i$ to 0 and solving for the remaining elements can evaluate a particular solution $\boldsymbol{v}$. Then the constant c can be computed by substituting Eq. (8) into Eq. (7):

$$c = -\boldsymbol{x}_i^T\boldsymbol{Mv} - \frac{1}{2}\boldsymbol{x}_i^T\frac{\partial \boldsymbol{M}}{\partial b}\boldsymbol{x}_i \tag{9}$$

Premultiplying Eq. (6) by $\boldsymbol{x}_i^T$, the design derivatives of the eigenvalue are given as

$$\frac{d\lambda_i}{db} = \boldsymbol{x}_i^T\left[\frac{\partial \boldsymbol{K}}{\partial b} - \lambda_i\frac{\partial \boldsymbol{M}}{\partial b}\right]\boldsymbol{x}_i \tag{10}$$

It is important to note that Nelson's method works only for the derivatives of distinct eigenvalues.

B. Dailey's Method

In the case of multiple eigenvalues, because the eigenspace spanned by the eigenvectors corresponding to the multiple eigenvalue is degenerate, any linear combination of eigenvectors can be an associated eigenvector. To find the eigenvector derivatives, the new eigenvector must be evaluated first, which lies adjacent to the m distinct eigenvectors.

To find the derivatives of the multiple eigenvalues and their associated eigenvectors, we differentiate Eq. (5) with respect to the design variable b:

$$(\boldsymbol{K} - \lambda\boldsymbol{M})\frac{d\boldsymbol{X}}{db} = \left[\lambda\frac{\partial \boldsymbol{M}}{\partial b} - \frac{\partial \boldsymbol{K}}{\partial b}\right]\boldsymbol{X} + \boldsymbol{MX}\frac{d\boldsymbol{\Lambda}}{db} \tag{11}$$

where the rank of the coefficient matrix $(\boldsymbol{K} - \lambda\boldsymbol{M})$ becomes $(n\text{-}m)$. Recall that $\lambda \in R$ is an eigenvalue of multiplicity m.

Premultiplication of Eq. (11) by $\boldsymbol{X}^T$ yields the eigenvalue derivatives as

$$\frac{d\boldsymbol{\Lambda}}{db} = \boldsymbol{X}^T\left[\frac{\partial \boldsymbol{K}}{\partial b} - \lambda\frac{\partial \boldsymbol{M}}{\partial b}\right]\boldsymbol{X} \tag{12}$$

where M orthonormality of Eq. (4) is used.

If $\boldsymbol{V}$ is any solution to Eq. (11) then $\boldsymbol{V} + \boldsymbol{XC}$ is also a solution, where $\boldsymbol{C} \in R^{m\times m}$ is the constant matrix to be determined later. The diagonal elements of $\boldsymbol{C}$ is obtained by substituting $\boldsymbol{V} + \boldsymbol{XC}$ into the design derivatives of Eq. (4):

$$\boldsymbol{C} + \boldsymbol{C}^T = -\boldsymbol{V}^T\boldsymbol{MX} - \boldsymbol{X}^T\boldsymbol{MV} - \boldsymbol{X}^T\frac{\partial \boldsymbol{M}}{\partial b}\boldsymbol{X} \tag{13}$$

The off-diagonal elements of $\boldsymbol{C}$ can be determined by differentiating Eq. (5) twice as follows [12]:

$$\begin{aligned}\boldsymbol{C}\frac{d\boldsymbol{\Lambda}}{db} - \frac{d\boldsymbol{\Lambda}}{db}\boldsymbol{C} + \frac{1}{2}\frac{d^2\boldsymbol{\Lambda}}{db^2} &= \boldsymbol{X}^T\left(\frac{\partial \boldsymbol{K}}{\partial b} - \lambda\frac{\partial \boldsymbol{M}}{\partial b}\right)\boldsymbol{V} \\ &- \boldsymbol{X}^T\left(\frac{\partial \boldsymbol{M}}{\partial b}\boldsymbol{X} + \boldsymbol{MV}\right)\frac{d\boldsymbol{\Lambda}}{db} + \frac{1}{2}\boldsymbol{X}^T\left(\frac{\partial^2 \boldsymbol{K}}{\partial b^2} - \lambda\frac{\partial^2 \boldsymbol{M}}{\partial b^2}\right)\boldsymbol{X}\end{aligned} \tag{14}$$

Because $d^2\boldsymbol{\Lambda}/db^2$ are diagonal and $[\boldsymbol{C}(d\boldsymbol{\Lambda}/db) - (d\boldsymbol{\Lambda}/db)\boldsymbol{C}]$ have zeros on the diagonal, we can obtain the off-diagonal elements of $\boldsymbol{C}$. Thus the design derivatives of eigenvectors are given as

$$\frac{d\boldsymbol{X}}{db} = \boldsymbol{V} + \boldsymbol{XC} \tag{15}$$

It is important to note that the second order derivatives of stiffness and mass matrices are necessary to evaluate the design derivatives of eigenvectors as seen in Eq. (14).

IV. Adjoint Method for Eigenvalue Problem

A. Derivatives of Distinct Eigenvalue and Its Eigenvector

Consider a general response function of an eigenvalue problem represented in terms of eigenvalues, eigenvectors, and a design variable as follows:

$$g = g(\lambda_i, \boldsymbol{x}_i, b) \tag{16}$$

where λ_i and $\boldsymbol{x}_i$ are the ith eigenvalue and its associate eigenvector, respectively. It is assumed that the response function whose design sensitivity needs to be evaluated is continuous and differentiable with respect to its arguments.

To develop the adjoint method for a distinct eigenvalue, we first define the augmented function using the response function of Eq. (16) and eigenvalue equations given in Eqs. (1) and (4) as [9]

$$L = g(\lambda_i, \boldsymbol{x}_i, b) + \boldsymbol{u}^T(\boldsymbol{K} - \lambda_i\boldsymbol{M})\boldsymbol{x}_i + \tfrac{1}{2}v\left(1 - \boldsymbol{x}_i^T\boldsymbol{Mx}_i\right) \tag{17}$$

where $\boldsymbol{u} \in R^n$ and $v \in R$ are the adjoint variables, which will be determined later.

To construct the adjoint equation, the total design derivative of the response function is represented as an explicit design derivative of the augmented function given in Eq. (17), that is,

$$\frac{dg}{db}=\frac{\partial L}{\partial b}=\frac{\partial g}{\partial b}+\boldsymbol{u}^T\left(\frac{\partial \boldsymbol{K}}{\partial b}-\lambda_i\frac{\partial \boldsymbol{M}}{\partial b}\right)\boldsymbol{x}_i-\frac{1}{2}\upsilon\boldsymbol{x}_i^T\frac{\partial \boldsymbol{M}}{\partial b}\boldsymbol{x}_i \tag{18}$$

Now the adjoint equation can be obtained by requiring the implicit design derivative of the augmented function to vanish. This leads to the following equation:

$$0=\frac{\partial g}{\partial \lambda_i}\frac{d\lambda_i}{db}+\left(\frac{\partial g}{\partial \boldsymbol{x}_i}\right)^T\frac{d\boldsymbol{x}_i}{db}-\frac{d\lambda_i}{db}\boldsymbol{u}^T\boldsymbol{M}\boldsymbol{x}_i+\boldsymbol{u}^T(\boldsymbol{K}-\lambda_i\boldsymbol{M})\frac{d\boldsymbol{x}_i}{db}$$
$$-\upsilon\boldsymbol{x}_i^T\boldsymbol{M}\frac{d\boldsymbol{x}_i}{db}\quad \text{for } \forall\ \frac{d\lambda_i}{db},\frac{d\boldsymbol{x}_i}{db} \tag{19}$$

Rearranging Eq. (19), we have

$$0=\frac{d\lambda_i}{db}\left(\frac{\partial g}{\partial \lambda_i}-\boldsymbol{x}_i^T\boldsymbol{M}\boldsymbol{u}\right)+\left(\frac{d\boldsymbol{x}_i}{db}\right)^T\left(\frac{\partial g}{\partial \boldsymbol{x}_i}+[\boldsymbol{K}-\lambda_i\boldsymbol{M}]\boldsymbol{u}-\upsilon\boldsymbol{M}\boldsymbol{x}_i\right)$$
$$\text{for } \forall\ \frac{d\lambda_i}{db},\frac{d\boldsymbol{x}_i}{db} \tag{20}$$

where the fact that $\boldsymbol{K}$ and $\boldsymbol{M}$ are symmetric matrices in structural analysis is employed. Because Eq. (20) holds for arbitrary $d\lambda_i/db$ and $d\boldsymbol{x}_i/db$, we have the adjoint equations as follows:

$$\begin{bmatrix}\boldsymbol{K}-\lambda_i\boldsymbol{M} & -\boldsymbol{M}\boldsymbol{x}_i\\ -\boldsymbol{x}_i^T\boldsymbol{M} & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{u}\\ \upsilon\end{bmatrix}=\begin{bmatrix}-\frac{\partial g}{\partial \boldsymbol{x}_i}\\ -\frac{\partial g}{\partial \lambda_i}\end{bmatrix} \tag{21}$$

It can be observed from Eq. (21) that the coefficient matrix is nonsingular if the eigenvalues are distinct and composed of the stiffness and mass matrices and the eigenvalue and eigenvectors of the ith mode only. Nonsingularity of the coefficient matrix is proved in Appendix A. The right-hand side of Eq. (21), pseudoload, can be given explicitly from the response function.

To find the design derivatives of the ith eigenvalue and its associated eigenvector, the response function is chosen for the eigenvector and its associated eigenvalue as follows:

$$\boldsymbol{g}=[\boldsymbol{x}_i^T\quad \lambda_i]^T \tag{22}$$

Then pseudoload for the jth element of the ith eigenvector $\boldsymbol{x}_i$ becomes

$$\boldsymbol{p}_{x_j}=[0\ \cdots\ 0\ -1\ 0\ \cdots\ 0]^T;\qquad j=1,\ldots,n \tag{23}$$

where the jth element of $\boldsymbol{p}_{x_j}\in R^{n+1}$ is -1 and other elements are all zeros. The pseudoload for the ith eigenvector λ_i becomes

$$\boldsymbol{p}_\lambda=[0\ \cdots\ 0\ -1]^T \tag{24}$$

Let us denote the adjoint variables by $[\boldsymbol{u}_j^T\quad \upsilon_j]^T$ corresponding to the jth element of the ith eigenvector and $[\boldsymbol{u}_{n+1}^T\quad \upsilon_{n+1}]^T$ corresponding to the ith eigenvalue. That is, the adjoint variables for the ith eigenvectors and eigenvalue can be represented as

$$\begin{bmatrix}\boldsymbol{U}\\ \boldsymbol{\upsilon}^T\end{bmatrix}\equiv\begin{bmatrix}\boldsymbol{u}_1 & \boldsymbol{u}_2 & \cdots & \boldsymbol{u}_{n+1}\\ \upsilon_1 & \upsilon_2 & \cdots & \upsilon_{n+1}\end{bmatrix} \tag{25}$$

Then the adjoint equation of Eq. (21) becomes

$$\begin{bmatrix}\boldsymbol{K}-\lambda_i\boldsymbol{M} & -\boldsymbol{M}\boldsymbol{x}_i\\ -\boldsymbol{x}_i^T\boldsymbol{M} & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{U}\\ \boldsymbol{\upsilon}^T\end{bmatrix}=[\boldsymbol{p}_{x_1}\quad \boldsymbol{p}_{x_2}\quad \cdots\quad \boldsymbol{p}_{x_n}\quad \boldsymbol{p}_\lambda]=-\boldsymbol{I} \tag{26}$$

The right-hand side of the adjoint equation becomes identity matrix. Once we solve the system equation of Eq. (26), we substitute the adjoint variables into Eq. (18) to obtain the design sensitivity coefficients of the eigenvalues and eigenvectors:

$$\frac{d\boldsymbol{g}}{db}=\begin{bmatrix}d\boldsymbol{x}_i/db\\ d\lambda_i/db\end{bmatrix}=\boldsymbol{U}^T\left(\frac{\partial \boldsymbol{K}}{\partial b}-\lambda_i\frac{\partial \boldsymbol{M}}{\partial b}\right)\boldsymbol{x}_i-\frac{1}{2}\boldsymbol{\upsilon}\boldsymbol{x}_i^T\frac{\partial \boldsymbol{M}}{\partial b}\boldsymbol{x}_i \tag{27}$$

It is very important to note that the adjoint variables can be evaluated by using the eigenvalue and corresponding eigenvectors of the mode being differentiated only.

B. Derivatives of Multiple Eigenvalues and Their Eigenvectors

To derive the adjoint equation for the design derivatives of multiple eigenvalues and their associated eigenvectors, we assume that the design derivatives of the multiple eigenvalue will separate such that the eigenvalue derivative is denoted by

$$\frac{d\boldsymbol{\Lambda}}{db}=\mathrm{diag}\left(\frac{d\lambda_1}{db},\frac{d\lambda_2}{db},\ldots,\frac{d\lambda_m}{db}\right) \tag{28}$$

Now let us define the response functions, $\boldsymbol{G}\in R^{(m+n)\times m}$, to evaluate the derivatives of the multiple eigenvalues and their associated eigenvectors as follows:

$$\boldsymbol{G}=\begin{bmatrix}\boldsymbol{X}\\ \boldsymbol{\Lambda}\end{bmatrix}=\begin{bmatrix}\boldsymbol{x}_1 & \boldsymbol{x}_2 & \cdots & \boldsymbol{x}_m\\ \lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_m\end{bmatrix} \tag{29}$$

where $\lambda_1=\lambda_2=\cdots=\lambda_m$. Then the augmented function can be defined as follows:

$$\boldsymbol{L}=\boldsymbol{G}+\boldsymbol{U}^T(\boldsymbol{K}\boldsymbol{X}-\boldsymbol{M}\boldsymbol{X}\boldsymbol{\Lambda})+\tfrac{1}{2}\boldsymbol{V}^T(\boldsymbol{I}-\boldsymbol{X}^T\boldsymbol{M}\boldsymbol{X}) \tag{30}$$

where $\boldsymbol{U}\in R^{n\times(m+n)}$ and $\boldsymbol{V}\in R^{m\times(m+n)}$ denote the adjoint variables that will be determined later. Thus design derivatives of multiple eigenvalues and their associated eigenvectors are given by expanding Eq. (27) as follows:

$$\frac{d\boldsymbol{G}}{db}=\boldsymbol{U}^T\left(\frac{\partial \boldsymbol{K}}{\partial b}\boldsymbol{X}-\frac{\partial \boldsymbol{M}}{\partial b}\boldsymbol{X}\boldsymbol{\Lambda}\right)-\frac{1}{2}\boldsymbol{V}^T\boldsymbol{X}^T\frac{\partial \boldsymbol{M}}{\partial b}\boldsymbol{X} \tag{31}$$

Requiring the implicit design derivative of the augmented function to vanish leads

$$0=\left(\frac{d\Lambda}{db}\right)^T\left(\frac{\partial \boldsymbol{G}}{\partial \Lambda}-\boldsymbol{X}^T\boldsymbol{M}\boldsymbol{U}\right)$$
$$+\left(\frac{d\boldsymbol{X}}{db}\right)^T\left(\frac{\partial \boldsymbol{G}}{\partial \boldsymbol{X}}+[\boldsymbol{K}-\lambda\boldsymbol{M}]\boldsymbol{U}-\boldsymbol{M}\boldsymbol{X}\boldsymbol{V}\right) \tag{32}$$

for $\forall\ d\Lambda/db$, $d\boldsymbol{X}/db$. Therefore we have the adjoint equation as follows:

$$\begin{bmatrix}\boldsymbol{K}-\lambda\boldsymbol{M} & -\boldsymbol{M}\boldsymbol{X}\\ -\boldsymbol{X}^T\boldsymbol{M} & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{U}\\ \boldsymbol{V}\end{bmatrix}=-\boldsymbol{I} \tag{33}$$

Substituting Eq. (33) into Eq. (31), we have

$$\frac{d\boldsymbol{G}}{db}=\begin{bmatrix}d\boldsymbol{X}/db\\ d\boldsymbol{\Lambda}/db\end{bmatrix} \tag{34}$$

Note that the coefficients matrix of Eq. (33) is nonsingular if the derivatives of the multiple eigenvalues are distinct. Nonsingularity of the coefficient matrix is analytically proved in Appendix A.

The proposed adjoint method for design sensitivity of multiple eigenvectors can be computed by using the multiple eigenvalues and their associated eigenvectors of the mode being differentiated without linear combination of other eigenvectors or the second-order derivative of the eigenvalue problem as Dailey's method. Moreover it can be observed from Eq. (33) that the adjoint equation is independent of the number of design variables, that is, the adjoint method requires the solution of adjoint equation once regardless of the number of design variables. However Dailey's method requires the solutions to the system equations of Eq. (11) as many as the number of design variables. Therefore the proposed adjoint method will be more efficient as the number of design variables is larger than 2.

V. Examples

An analytical example of a two-degree-of-freedom spring-mass system, planar grillage structure, and a finite element example of cantilever beam are here illustrated to calculate the design sensitivity coefficients of multiple eigenvalues and their associated eigenvectors. Design sensitivity coefficients of the example are compared with those evaluated by using exact differentiation or the central finite difference method. Throughout the finite element example, eigenvalues, eigenvectors, and adjoint variables are computed by MATLAB [19].

A. Two-Degree-of-Freedom Spring-Mass System

Let us consider a vibration of two-degree-of-freedom spring-mass system as given in Fig. 1 whose eigenvalue problem is defined as [10]

$$\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} m_1+m_2 & -m_2 \\ -m_2 & m_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{35}$$

The eigenvalues of Eq. (35) are

$$\boldsymbol{\Lambda} = \begin{bmatrix} k_1/m_1 & 0 \\ 0 & k_2/m_2 \end{bmatrix} \tag{36}$$

And the M-orthonormalized eigenvectors are given as

$$\boldsymbol{X} = \begin{bmatrix} 1/\sqrt{m_1} & 0 \\ 1/\sqrt{m_1} & -1/\sqrt{m_2} \end{bmatrix} \tag{37}$$

For the case when $k_1 = k_2 = k$, $m_1 = m_2 = m$, the system has the multiple eigenvalues. Now consider the design sensitivities of the multiple eigenvalues and their associated eigenvectors, where the design variable is chosen as $b = m_2$.

The adjoint equation and adjoint variables are

$$\sqrt{m}\begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{U} \\ \boldsymbol{V} \end{bmatrix} = -\boldsymbol{I} \tag{38}$$

$$\begin{bmatrix} \boldsymbol{U} \\ \boldsymbol{V} \end{bmatrix} = \frac{1}{\sqrt{m}}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \tag{39}$$

Thus the design derivatives of eigenvalues and eigenvectors are given as

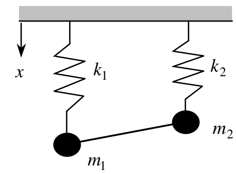

Fig. 1 Vibration of the two-degree-of-freedom spring-mass system.

$$\begin{aligned} \frac{d\boldsymbol{G}}{db} &= -\boldsymbol{U}^T\frac{\partial \boldsymbol{M}}{\partial b}\boldsymbol{X}\boldsymbol{\Lambda} - \frac{1}{2}\boldsymbol{V}^T\boldsymbol{X}^T\frac{\partial \boldsymbol{M}}{\partial b}\boldsymbol{X} \\ &= -\frac{1}{m}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}\left(\frac{k}{m}\right) \\ &\quad - \frac{1}{2}\frac{1}{m\sqrt{m}}\begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \\ &= \frac{1}{2m}\begin{bmatrix} 0 & 0 \\ 0 & 1/\sqrt{m} \\ 0 & 0 \\ 0 & -2k/m \end{bmatrix} \end{aligned} \tag{40}$$

Therefore we have design derivatives of the multiple eigenvalues and their associated eigenvectors as follows:

$$\frac{d\boldsymbol{\Lambda}}{db} = \begin{bmatrix} 0 & 0 \\ 0 & -k/m^2 \end{bmatrix} \tag{41}$$

$$\frac{d\boldsymbol{X}}{db} = \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & 1/m^{3/2} \end{bmatrix} \tag{42}$$

which are identical to the exact design derivatives of the analytical eigenvalues and eigenvectors in Eqs. (36) and (37) with respect to m_2, respectively.

B. Planar Grillage Structure

Let us consider a three-degree-of-freedom planar grillage structure that is composed with four identical members and modeled by two-dimensional beam elements as shown in Fig. 2 [11]. Thus the nodal displacement vector is chosen as

$$\boldsymbol{q} = [x \quad y \quad \theta_z]^T \tag{43}$$

The global stiffness and mass matrices are of the form

$$\boldsymbol{K} = E\begin{bmatrix} \frac{A_1+A_3}{L} + 12\frac{I_2+I_4}{L^3} & 0 & -6\frac{I_2-I_4}{L^2} \\ & 12\frac{I_1+I_3}{L^3} + \frac{A_2+A_4}{L} & 6\frac{I_1-I_3}{L^2} \\ \text{sym} & & 4\frac{I_1+I_2+I_3+I_4}{L} \end{bmatrix} \tag{44}$$

$$\boldsymbol{M} = \rho\begin{bmatrix} \frac{1}{3}(A_1+A_3)L + \frac{13}{35}(A_2+A_4)L & 0 & -\frac{11}{210}(A_2-A_4)L^2 \\ & \frac{13}{35}(A_1+A_3)L + \frac{1}{3}(A_2+A_4)L & \frac{11}{210}(A_1-A_4)L^2 \\ \text{sym} & & \frac{1}{105}(A_1+A_2+A_3+A_4)L^3 \end{bmatrix} \tag{45}$$

Material properties and geometric parameters are given in Table 1. The eigenvalues and their associated M-orthogonal eigenvectors are presented in Table 2. Note that second and third eigenvalues are multiple.

Now consider the design sensitivities of eigenvalues and their associated eigenvectors in which design variable is chosen as the height of the member 1, that is, $b = h_1 = 0.01(m)$. Then design derivatives of stiffness and mass matrices expressed in Eqs. (41) and (42) are given as

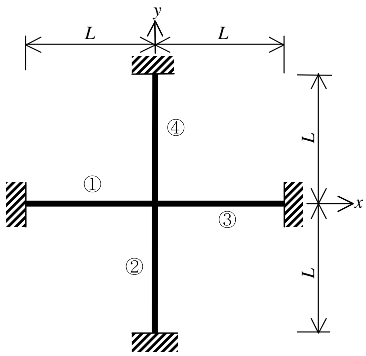

Fig. 2 Three degree-of freedom planar grillage structure.

$$\frac{d\boldsymbol{K}}{db} = E\begin{bmatrix} \frac{1}{L}\frac{dA_1}{db} & 0 & 0 \\ & \frac{12}{L^3}\frac{dI_1}{db} & \frac{6}{L^2}\frac{dI_1}{db} \\ \text{sym} & & \frac{4}{L}\frac{dI_1}{db} \end{bmatrix} \tag{46}$$

$$\frac{d\boldsymbol{M}}{db} = \rho\begin{bmatrix} \frac{1}{3}L\frac{dA_1}{db} & 0 & 0 \\ & \frac{13}{35}L\frac{dA_1}{db} & \frac{11}{210}L^2\frac{dA_1}{db} \\ \text{sym} & & \frac{1}{105}L^2\frac{dA_1}{db} \end{bmatrix} \tag{47}$$

where $dA_1/db = w$ and $dI_1/db = wh_1^2/4$.

Design sensitivities of eigenvalue and their associated eigenvectors are evaluated by using the adjoint method listed in Table 3. We can observe that these results are identical to those obtained by the central finite difference method with 0.1% design perturbation.

C. Three-Dimensional Beam Vibration

Let us consider a simple finite element model for a three-dimensional cantilever beam with square cross section as shown in Fig. 3. When torsional and axial deformations are assumed to be negligible, global stiffness and mass matrices with built-in boundary condition are of the form

$$\boldsymbol{K} = \frac{E}{L}\begin{bmatrix} \frac{24I_z}{L^2} & 0 & 0 & 0 & -\frac{12I_z}{L^2} & 0 & 0 & \frac{6I_z}{L} \\ & \frac{24I_y}{L^2} & 0 & 0 & 0 & -\frac{12I_y}{L^2} & -\frac{6I_y}{L} & 0 \\ & & 8I_y & 0 & 0 & \frac{6I_y}{L} & 2I_y & 0 \\ & & & 8I_z & -\frac{6I_z}{L} & 0 & 0 & 2I_z \\ & & & & \frac{12I_z}{L^2} & 0 & 0 & -\frac{6I_z}{L} \\ & & & & & \frac{12I_y}{L^2} & \frac{6I_y}{L} & 0 \\ & & & & & & 4I_y & 0 \\ \text{sym} & & & & & & & 4I_z \end{bmatrix} \tag{48}$$

Table 1 Material properties and geometric parameters of planar grillage structure

Elastic modulus	300 GPa
Density	74 kg/m^3
L	0.72 m
w, width of beam	0.01 m
h, height of beam	0.01 m

Table 2 Eigenvalues and their associated eigenvectors of planar grillage structure

	1st mode	2nd mode	3rd mode
λ	5.2799×10^4	1.1099×10^7	1.1099×10^7
	0	0	0.3647
$\boldsymbol{x}$	0	0.3647	0
	3.0828	0	0

Table 3 Design sensitivities of eigenvalues and eigenvectors of planar grillage structure

	1st mode	2nd mode	3rd mode
$d\lambda/db$	2.6400×10^6	-2.9214×10^8	2.9235×10^8
	0	0	−4.3148
$d\boldsymbol{x}/db$	7.1689×10^{-3}	−4.8079	0
	−38.5354	−69.7476	0

$$\boldsymbol{M} = \frac{\rho AL}{420}\begin{bmatrix} 312 & 0 & 0 & 0 & 54 & 0 & 0 & -13L \\ & 312 & 0 & 0 & 0 & 54 & 13L & 0 \\ & & 8L^2 & 0 & 0 & -13L & -3L^2 & 0 \\ & & & 8L^2 & 13L & 0 & 0 & -3L^2 \\ & & & & 156 & 0 & 0 & -22L \\ & & & & & 156 & 22L & 0 \\ & & & & & & 4L^2 & 0 \\ \text{sym} & & & & & & & 4L^2 \end{bmatrix} \tag{49}$$

The nodal displacement vector is of the form

$$\boldsymbol{q} = [\,y_2 \quad z_2 \quad \theta_{y2} \quad \theta_{z2} \quad y_3 \quad z_3 \quad \theta_{y3} \quad \theta_{z3}\,]^T \tag{50}$$

Choosing design variable as the z-axis area moment of the element 2, that is, $b = I_z$, and differentiating Eqs. (48) and (49) with respect to the design variable, we have

$$\frac{\partial \boldsymbol{M}}{\partial b} = \boldsymbol{0} \tag{51}$$

And

$$\frac{\partial \boldsymbol{K}}{\partial b} = \frac{E}{L}\begin{bmatrix} \frac{12}{L^2} & 0 & 0 & \frac{6}{L} & -\frac{12}{L^2} & 0 & 0 & \frac{6}{L} \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 4 & -\frac{6}{L} & 0 & 0 & 2 \\ & & & & \frac{12}{L^2} & 0 & 0 & -\frac{6}{L} \\ & & & & & 0 & 0 & 0 \\ & & & & & & 0 & 0 \\ \text{sym} & & & & & & & 4 \end{bmatrix} \tag{52}$$

For computational simplicity, let us assume the nondimensional geometric and material properties as follows:

$$L = 1 \qquad I_y = I_z = 1 \qquad \rho = 1 \qquad A = 420 \qquad E = 1000 \tag{53}$$

Then the lowest multiple eigenvalues and their associated M-orthonomalized eigenvectors are

$$\lambda_1 = \lambda_2 = 1.84142 \tag{54}$$

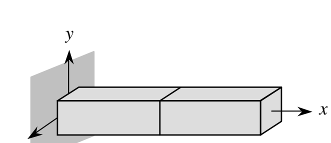

Fig. 3 Two-element cantilever beam with multiple eigenvalues.

Table 4 Design sensitivity coefficients of the first repeated eigenvalue and its associated eigenvectors, and their comparison with the central finite difference method

	Adjoint method	R, %
$d\lambda_1/db$	0.091856	100.01
$d\boldsymbol{x}_1/db$	−0.001200	100.01
	0	100.00
	0	100.00
	−0.001999	100.01
	0.002052	100.01
	0	100.00
	0	100.00
	0.005109	100.01

$$\boldsymbol{X} = [\boldsymbol{x}_1 \quad \boldsymbol{x}_2] = \begin{bmatrix} -0.0235 & 0 \\ 0 & 0.0235 \\ 0 & -0.0402 \\ -0.0402 & 0 \\ -0.0691 & 0 \\ 0 & 0.0691 \\ 0 & -0.0475 \\ -0.0475 & 0 \end{bmatrix} \tag{55}$$

It is important to note that the beam's modes are repeated one by one because of the geometric symmetry. The z-axis component's design derivatives of the lowest eigenvalue and its associated eigenvectors are given in Table 4. However the y-axis component's derivatives of the lowest eigenvalue and its eigenvectors are zero. To show the accuracy of the developed method, we give the ratio R between the design sensitivity coefficients predicted by the adjoint method dg/db and those by the central finite difference method $\Delta g/\Delta b$ in the presenting results as follows:

$$R = \frac{\Delta g/\Delta b}{dg/db} \times 100(\%) \tag{56}$$

$$\Delta g = [g(b + \Delta b) - g(b - \Delta b)]/2 \tag{57}$$

where the design perturbation in the central finite difference method is assumed to be $\Delta b = 0.01 \times b$. A ratio of 100% means that the predicted design sensitivity coefficient matches exactly with that computed by the finite difference method.

The sensitivity coefficients of the multiple eigenvalues and their eigenvectors are in very good agreement with those computed by using the central finite difference method within 0.01% as observed in Table 4.

VI. Conclusions

Design sensitivity analysis of the multiple eigenvalue problem based on the adjoint method is presented for the real symmetric structural eigenvalue problem. The adjoint equation for the sensitivity analysis of eigenvalues and eigenvectors is obtained by requiring the implicit design variations of the augmented function to vanish. Once we calculate the adjoint variables from the linear system equation, we can evaluate the design sensitivity coefficients of the distinct and multiple eigenvalues and eigenvectors. An analytical example and numerical examples are demonstrated to calculate the design sensitivity coefficients of the multiple eigenvalue problems and to verify the accuracy and efficiency of the developed method.

Based on the study on design sensitivity analysis of the multiple eigenvalue problem, we can conclude the following:

1) A true adjoint method for design sensitivity analysis of the distinct and multiple eigenvalues problem is proposed.

2) The proposed adjoint method for design sensitivity of an eigenvector can be computed by using the eigenvalue and its associated eigenvectors being differentiated without a linear combination of some eigenvectors or the second-order derivative of the eigenvalue problem.

3) The adjoint method requires the solution of the adjoint equation once regardless of the number of design variables.

4) For simple structures, design sensitivity of distinct and multiple eigenvalues and their associate eigenvectors are evaluated by using the developed adjoint method and their accuracies are verified with those obtained by analytical or the central finite difference method.

Note that the suggested method is applicable only to the design derivative of multiple eigenvalues and their associated eigenvectors whose derivatives become distinct. The proposed adjoint method can be easily implemented outside of a commercial finite element analysis program.

Appendix A: Proof of Nonsingularity of the Coefficient Matrix in Eq. (33)

Consider the coefficient matrix of the system equation:

$$\begin{bmatrix} \boldsymbol{K} - \lambda \boldsymbol{M} & -\boldsymbol{M}\boldsymbol{X} \\ -\boldsymbol{X}^T\boldsymbol{M} & \boldsymbol{0} \end{bmatrix}_{(m+n)\times(m+n)} \begin{bmatrix} \boldsymbol{U} \\ \boldsymbol{V} \end{bmatrix}_{(m+n)\times(m+n)} = -\boldsymbol{I}_{(m+n)\times(m+n)} \tag{A1}$$

$$\bar{\boldsymbol{K}} \equiv \begin{bmatrix} \boldsymbol{K} - \lambda \boldsymbol{M} & -\boldsymbol{M}\boldsymbol{X} \\ -\boldsymbol{X}^T\boldsymbol{M} & \boldsymbol{0} \end{bmatrix} \tag{A2}$$

To prove that the coefficient matrix of Eq. (A2) is nonsingular, we introduce an arbitrary nonsingular matrix $\boldsymbol{P} \in R^{(n+m)\times(n+m)}$. Let us assume that the arbitrary nonsingular square matrix $\boldsymbol{P}$ is of the form as follows:

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{\Phi} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_m \end{bmatrix} \tag{A3}$$

The submatrix $\boldsymbol{\Phi} \in R^{n\times n}$ is assumed as

$$\boldsymbol{\Phi} = [\boldsymbol{\Phi}_1 \quad \boldsymbol{\Phi}_2 \quad \cdots \quad \boldsymbol{\Phi}_{n-m} \quad \boldsymbol{x}_1 \quad \boldsymbol{x}_2 \quad \cdots \quad \boldsymbol{x}_m] \tag{A4}$$

where $\boldsymbol{\Phi}_j \in R^n$ are selected arbitrary independent vectors chosen to be independent from $\boldsymbol{X} = [\boldsymbol{x}_1 \quad \boldsymbol{x}_2 \quad \cdots \quad \boldsymbol{x}_m]$, eigenvectors of the original multiple eigenproblem given in Eq. (5), such that $\boldsymbol{P}$ becomes nonsingular. Now, pre- and postmultiplying by $\boldsymbol{P}^T$ and $\boldsymbol{P}$ to $\bar{\boldsymbol{K}}$ yields

$$\boldsymbol{P}^T\bar{\boldsymbol{K}}\boldsymbol{P} = \begin{bmatrix} \boldsymbol{\Phi} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_m \end{bmatrix}^T \begin{bmatrix} \boldsymbol{K} - \lambda \boldsymbol{M} & -\boldsymbol{M}\boldsymbol{X} \\ -\boldsymbol{X}^T\boldsymbol{M} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Phi} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_m \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}^T[\boldsymbol{K} - \lambda \boldsymbol{M}]\boldsymbol{\Phi} & -\boldsymbol{\Phi}^T\boldsymbol{M}\boldsymbol{X} \\ -\boldsymbol{X}^T\boldsymbol{M}\boldsymbol{\Phi} & \boldsymbol{0} \end{bmatrix} \tag{A5}$$

Because λ is an m-multiple eigenvalue, the rank of $[\boldsymbol{K} - \lambda\boldsymbol{M}]$ is (n-m) such that the last m columns and rows of $\boldsymbol{\Phi}^T[\boldsymbol{K} - \lambda\boldsymbol{M}]\boldsymbol{\Phi}$ have all zeros. Then a submatrix of Eq. (A5) can be rewritten as

$$\boldsymbol{\Phi}^T[\boldsymbol{K} - \lambda \boldsymbol{M}]\boldsymbol{\Phi} = \begin{bmatrix} \hat{\boldsymbol{K}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0}_{m\times m} \end{bmatrix} \tag{A6}$$

Note that $\text{rank}(\hat{\boldsymbol{K}}) = n - m$. Another submatrix of Eq. (A5) can be written as

$$\boldsymbol{\Phi}^T\boldsymbol{M}\boldsymbol{X} = \begin{bmatrix} \hat{\boldsymbol{B}} \\ \boldsymbol{I}_m \end{bmatrix} \quad \text{and} \quad \boldsymbol{X}^T\boldsymbol{M}\boldsymbol{\Phi} = [\hat{\boldsymbol{B}}^T \quad \boldsymbol{I}_m] \tag{A7}$$

where $\hat{\boldsymbol{B}} \in R^{n\times n}$ is a nonzero matrix and the M-orthonormality condition of Eq. (4) is applied in Eq. (A7). Thus we have

$$\boldsymbol{P}^T\bar{\boldsymbol{K}}\boldsymbol{P} = \begin{bmatrix} \hat{\boldsymbol{K}} & \boldsymbol{0} & \hat{\boldsymbol{B}} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_m \\ \hat{\boldsymbol{B}}^T & \boldsymbol{I}_m & \boldsymbol{0} \end{bmatrix} \tag{A8}$$

Determinant of the partitioned matrix is given as (see Appendix B)

$$
\begin{aligned}
|\boldsymbol{P}^T\bar{\boldsymbol{K}}\boldsymbol{P}| &= \begin{vmatrix} \hat{\boldsymbol{K}} & \boldsymbol{0} & \hat{\boldsymbol{B}} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_m \\ \hat{\boldsymbol{B}}^T & \boldsymbol{I}_m & \boldsymbol{0} \end{vmatrix} \\
&= \left| \hat{\boldsymbol{K}} - [\boldsymbol{0} \quad \hat{\boldsymbol{B}}] \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_m \\ \boldsymbol{I}_m & \boldsymbol{0} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{0} \\ \hat{\boldsymbol{B}}^T \end{bmatrix} \right| \begin{vmatrix} \boldsymbol{0} & \boldsymbol{I}_m \\ \boldsymbol{I}_m & \boldsymbol{0} \end{vmatrix} \\
&= (-1)^m |\hat{\boldsymbol{K}}| \neq 0
\end{aligned}
\tag{A9}
$$

Recall the property of determinant:

$$
|\boldsymbol{P}^T\bar{\boldsymbol{K}}\boldsymbol{P}| = |\boldsymbol{P}^T||\bar{\boldsymbol{K}}||\boldsymbol{P}| \neq 0 \tag{A10}
$$

Because we assumed that $\boldsymbol{P}$ is nonsingular, $|\bar{\boldsymbol{K}}| \neq 0$ from Eq. (A10). Therefore it is proved that the coefficient matrix $\bar{\boldsymbol{K}}$ is nonsingular. When $m = 1$, that is, for a distinct eigenvalue, Eq. (A9) holds. Thus the coefficient matrix of Eq. (21) is nonsingular too.

Appendix B: Determinant of Partitioned Matrix

If $\boldsymbol{A}$ and $\boldsymbol{D}$ are square matrix and $|\boldsymbol{D}| \neq 0$, then the partitioned matrix can be decomposed as follows:

$$
\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{B}\boldsymbol{D}^{-1} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{D}^{-1}\boldsymbol{C} & \boldsymbol{I} \end{bmatrix} \tag{B1}
$$

Then determinant of the partitioned matrix is evaluated as follows [20]

$$
\begin{aligned}
\begin{vmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{vmatrix} &= \begin{vmatrix} \boldsymbol{I} & \boldsymbol{B}\boldsymbol{D}^{-1} \\ \boldsymbol{0} & \boldsymbol{I} \end{vmatrix} \begin{vmatrix} \boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{D} \end{vmatrix} \begin{vmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{D}^{-1}\boldsymbol{C} & \boldsymbol{I} \end{vmatrix} \\
&= |\boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C}||\boldsymbol{D}|
\end{aligned}
\tag{B2}
$$

Note that

$$
\begin{vmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{D} \end{vmatrix} = |\boldsymbol{A}||\boldsymbol{D}|, \qquad \begin{vmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{C} & \boldsymbol{D} \end{vmatrix} = |\boldsymbol{A}||\boldsymbol{D}| \tag{B3}
$$

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A. Berman
Associate Editor